Experimental and Lagrangian modeling of nonlinear water waves propagation on a sloping bottom

Meng-Syue Li, Yang-Yih Chen, Hung-Chu Hsu, A. Torres-Freyermuth

A R T I C L E   I N F O
Article history:
Received 10 July 2012
Accepted 12 January 2013

Keywords:
Lagrangian
Wave breaking
Sloping bottom
Experiment
Particle trajectory
Nonlinear water waves

A B S T R A C T
This paper presents an experimental and theoretical investigation of nonlinear water wave propagation over a sloping bed. Firstly, a series of monochromatic wave laboratory experiments were performed in order to measure the particle trajectories, evolution of wave profile, and wave phase velocity as wave propagates on a sloping bottom. The particle trajectories are quantified by means of images from a high speed camera, whereas the evolution of wave profile and variation of wave phase velocity are measured by a wave gauge array. Subsequently, the free-surface elevation, phase velocity, particle trajectories, and breaking wave height are estimated using a Lagrangian nonlinear wave transformation model. Model predictions show a reasonable agreement with experimental data.

1. Introduction

As water waves propagate onshore, the wave height increases and the wave profile becomes more skewed and asymmetric (Elgar and Guza, 1985) owing to nonlinear effects. The wave breaking occurs with the release of enormous energy and huge impact force, which will damage coastal structures such as breakwater and drive sediment transport along/across in the nearshore. Therefore, the wave breaking characteristics needs to be quantified for reliable prediction of sediment transport and structural design in coastal regions. Previous studies have discussed and proposed empirical formulas for estimating the breaking wave height as a function of deepwater wave height, wave steepness, and bottom slope (Tang, 1966; Iwagaki et al., 1974; Galvin, 1968; Goda, 1970, 1975, 2004, 2010; Sunamura and Horikawa, 1974; Sunamura, 1983; Rattanapitikon and Shibayanma, 2000, 2006). More recently, Deo and Jagdale (2003) used the neural networks in order to predict the breaking wave height and the water depth of wave breaking using experimental data. Tsai et al. (2005) found the previous empirical formulations where not able to reproduce the experimental results on steep beaches. Despite recent advances toward the parameterization of the wave breaking process, the major drawback in empirical wave breaking models is that they only provide the location of the breaking wave height, but can not describe the details of wave deformation (i.e., higher order statistics) and the related flow field.

The motion of a fluid particle within a propagating surface wave may be described by either observing the fluid velocity at a fixed position or the trajectory of a particle that is carried along with the flow. These two different descriptions are known as Eulerian and Lagrangian approach, respectively. For an incompressible fluid, the Eulerian approach is clearly preferred because of the corresponding continuity equation is linear. It is also well known that the Eulerian description for a free surface can always be expressed in Taylor series at a fixed water level, which implicitly assumes that the surface profile is a differentiable single-valued function. On the other hand, in the Lagrangian approach the surface elevation is specified through the positions of the surface particles. Unlike an Eulerian surface, which is given as an implicit function, a Lagrangian form is expressed through a parametric representation of particle motion. Hence, the Lagrangian description is more appropriate for limiting the free surface motion overcoming some limitation in the classical Eulerian solutions (Naciri and Mei, 1993; Buldakov et al., 2006; Chen and Hsu, 2009; Hsu et al., 2010).

Therefore, the aim of this work is to derive a theoretical formulation for predicting the breaking height and water depth using the nonlinear asymptotic solution in Lagrangian coordinates. The nonlinear asymptotic solution for surface waves propagating over a sloping bottom in Lagrangian description includes the effect of a sloping bottom and wave steepness. Using the perturbation method, Chen et al. (2012) derived a parametric expression of the
particle trajectories in terms of wave steepness and the bottom slope to a second-order. Thus, in order to describe the breaking wave mechanism, the kinematic stability parameter (K.S.P.) is applied and the breaking criterion \( u/C_w = 1 \), where \( C_w \) is wave phase velocity and \( u \) is the horizontal velocity of particle at the wave crest, is implemented for simulating wave breaking. A laboratory experiment is conducted in order to validate the theoretical results (free-surface wave elevation, phase velocity, particle trajectory and breaking wave height) on a sloping bottom.

2. Experimental setup and procedures

The purpose of this experiment is to quantitatively investigate the particle trajectories and wave phase velocity for progressive gravity waves on a sloping bottom. These experimental results can also be further employed to validate the theoretical results. Particle velocity can be verified during the process of comparing the orbits of water particle obtained by the theoretical solution and experimental measurements. Therefore, the breaking wave height and breaking water depth can be well predicted by the present theoretical solution. The experimental measurements were carried out in a glass-walled wave tank, 35.0 m \( \times \) 1.0 m \( \times \) 1.2 m, in the Department of Marine Environment and Engineering at National Sun Yat-sen University. A High-Speed camera was set up in front of the glass-wall at the breaking point location (i.e., \( x = -1 \) m). This method allows to successively capturing the particles motion near the wave breaking point. The wave gauge locations are given in Table 1. Wave gauges Nos. 17–22 corresponded to the group of wave gauge were moved depending on the position of the breaking point, where the No. 21 wave gauge corresponds to the breaking point location and the No. 22 wave gauge is behind the breaking point. The whole experimental reference frame is schematically shown in Fig. 1.

A piston-type wave generator was employed for generating monochromatic waves. The particles trajectories were visualized by means of spherical polystyrene (PS) beds of fluorescent red color of 0.1 cm diameter and a density near 1.000 g/cm\(^3\). Images were captured by a High-Speed camera (MS55k2, Canadian Photonic Labs Inc.), which has a 1280 \( \times \) 1080 pixel resolution and 1020 frames per second (fps) maximum framing rate. A transparent acrylic-plastic sheet (0.9 m \( \times \) 0.625 m), which was placed in the plane of the PS motion position, was calibrated at 1-mm intervals in 5 mm \( \times \) 5 mm grids. Its function is a virtual grid in the picture. According to Wang et al. (2012), the trajectory of the PS motion in the water waves could be inferred from the PS motion image data and virtual grid.

The experiments were conducted at a constant water depth \( (d=0.796 \) m) and different wave periods (\( T=0.802–1.41 \) s). The incident wave height \( H_0 \) was varied over a range of 0.0866–0.108 m. A summary of the experimental wave conditions is shown in Table 2. Moreover, the particle orbits for all conditions are shown in Table 3, where \( k_{0.0}X_0 \) is the dimensionless initial position of PS particle in \( x \)-direction and \( k_{0.0}Y_0 \) is the dimensionless initial position of PS particle in \( y \)-direction. The high-spatial resolution of free-surface elevation measurements allows a good estimate of the phase velocity.

### Table 1

<table>
<thead>
<tr>
<th>Number of wave gauge</th>
<th>Position (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 1</td>
<td>-11.92</td>
</tr>
<tr>
<td>No. 2</td>
<td>-9.92</td>
</tr>
<tr>
<td>No. 3</td>
<td>-7.805</td>
</tr>
<tr>
<td>No. 4</td>
<td>-7.795</td>
</tr>
<tr>
<td>No. 5</td>
<td>-6.844</td>
</tr>
<tr>
<td>No. 6</td>
<td>-6.829</td>
</tr>
<tr>
<td>No. 7</td>
<td>-5.848</td>
</tr>
<tr>
<td>No. 8</td>
<td>-5.83</td>
</tr>
<tr>
<td>No. 9</td>
<td>-4.866</td>
</tr>
<tr>
<td>No. 10</td>
<td>-4.851</td>
</tr>
<tr>
<td>No. 11</td>
<td>-3.866</td>
</tr>
<tr>
<td>No. 12</td>
<td>-3.851</td>
</tr>
<tr>
<td>No. 13</td>
<td>-2.924</td>
</tr>
<tr>
<td>No. 14</td>
<td>-2.912</td>
</tr>
<tr>
<td>No. 15</td>
<td>-2.109</td>
</tr>
<tr>
<td>No. 16</td>
<td>-2.101</td>
</tr>
<tr>
<td>No. 17</td>
<td>P17</td>
</tr>
<tr>
<td>No. 18</td>
<td>P17 +0.22</td>
</tr>
<tr>
<td>No. 19</td>
<td>P17 +0.354</td>
</tr>
<tr>
<td>No. 20</td>
<td>P17 +0.428</td>
</tr>
<tr>
<td>No. 21</td>
<td>P17 +0.463</td>
</tr>
<tr>
<td>No. 22</td>
<td>P17 +0.466</td>
</tr>
</tbody>
</table>

P17 is the position of No. 17 wave gauge.

### Table 2

<table>
<thead>
<tr>
<th>No.</th>
<th>( T ) (s)</th>
<th>( H_0 ) (m)</th>
<th>( d ) (m)</th>
<th>P17 (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.002</td>
<td>0.0907</td>
<td>0.796</td>
<td>-1.446</td>
</tr>
<tr>
<td>2</td>
<td>1.410</td>
<td>0.1086</td>
<td>0.796</td>
<td>-1.424</td>
</tr>
<tr>
<td>3</td>
<td>1.201</td>
<td>0.0865</td>
<td>0.796</td>
<td>-1.188</td>
</tr>
<tr>
<td>4</td>
<td>1.199</td>
<td>0.0989</td>
<td>0.796</td>
<td>-1.336</td>
</tr>
<tr>
<td>5</td>
<td>1.400</td>
<td>0.1066</td>
<td>0.796</td>
<td>-1.424</td>
</tr>
<tr>
<td>6</td>
<td>1.201</td>
<td>0.0940</td>
<td>0.796</td>
<td>-1.188</td>
</tr>
<tr>
<td>7</td>
<td>1.403</td>
<td>0.1001</td>
<td>0.796</td>
<td>-1.424</td>
</tr>
<tr>
<td>8</td>
<td>0.803</td>
<td>0.0900</td>
<td>0.796</td>
<td>-1.484</td>
</tr>
<tr>
<td>9</td>
<td>0.802</td>
<td>0.0904</td>
<td>0.796</td>
<td>-1.484</td>
</tr>
</tbody>
</table>

Fig. 1. Experimental frame and instruments setup.
A fluid particle is distinguished by the horizontal and vertical by tracing an individual fluid particle. For two-dimensional flow, the fluid motion is described by a set of trajectories $x(t)$ and $y(t)$, where $x$ and $y$ are Cartesian coordinates. The dependent variables, $x$ and $y$, indicate the position of any particle at time $t$ and are function of the independent variables $x_0$, $y_0$, and $t$. In a Lagrangian reference frame, the governing equations and boundary conditions for two-dimensional irrotational free-surface flow is given by Chen et al. (2012):

$$J = \frac{\partial (x,y)}{\partial (x_0,y_0)} = x_0 \frac{\partial y}{\partial x} - y_0 \frac{\partial x}{\partial y} = 1,$$  

(1)

$$x_0 \frac{\partial y}{\partial x} = \frac{\partial (x,y)}{\partial (x_0,y_0)} = x_0 \frac{\partial y}{\partial x} - y_0 \frac{\partial x}{\partial y} = 0,$$  

(2)

$$x_0 \frac{\partial x}{\partial y} = \frac{\partial (x,y)}{\partial (x_0,y_0)} = x_0 \frac{\partial x}{\partial y} - y_0 \frac{\partial y}{\partial x} = 0,$$  

(3)

$$\frac{\partial \phi}{\partial x_0} = x_0 \frac{\partial \phi}{\partial x} + y_0 \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial y_0} = x_0 \frac{\partial \phi}{\partial x} + y_0 \frac{\partial \phi}{\partial y},$$  

(4)

$$\frac{p}{\rho} = -\frac{\partial \phi}{\partial t} - g \frac{\partial y}{\partial x} + \frac{1}{2} (x_0^2 + y_0^2),$$  

(5)

in Eqs. (1)–(5), subscripts $x_0$, $y_0$, and $t$ denote partial differentiation with respect to the specified variable, $P(X_0,Y_0,t)$ is water pressure, and $\phi(x_0,y_0,t)$ is the velocity potential function in the Lagrangian system. Eq. (1) is the continuity equation and at $y_0 = 0$ to be the vertical label marked for a particle at free surface that set the invariant condition on the volume of a Lagrangian particle: Eq. (2) is the differentiation of Eq. (1) with respect to time. Eqs. (3) and (4) denote the irrotational flow condition and the corresponding Lagrangian velocity potential, respectively. Eq. (5) is the Bernoulli equation for the irrotational flow in Lagrangian description.

The wave motion has to satisfy a number of boundary conditions at the bottom and on the free water surface. On an immovable and impermeable sloping plane the no-flux boundary condition gives

$$y_t - 2y_x = 0, \quad y_x = y_u = d = 2x,$$  

(6)

and the dynamic boundary condition of zero pressure at the free surface is given by

$$P = 0, \quad y_0 = 0.$$

(7)

Furthermore, a stationary mass transport condition is required as waves propagate toward the beach. A horizontal hydrostatic pressure gradient, to balance the radiation stress of the progressive wave, will drive a return flow and a hence boundary condition should be imposed. This condition is necessary for the uniqueness of the solution and requires that at any cross-section of the $x$-$y$ plane, the mass transport should vanish:

$$y_t = 0, \quad y_x = 0.$$  

(8)

$$y_t = 0, \quad y_x = 0.$$  

(9)

The superscript $c$ express the physical quantity at $x = -\infty$. Because of the nonlinear effect, waves over constant depth induce a net flux of water. Thus, a constant depth streaming term is introduced in (9) which is adjusted by a unit function $U(z)$ in order to ensure that it can be reduced to the constant depth condition when the bottom slope is equal to zero.

3.2. Asymptotic solutions

To solve Eqs. (1)–(9), it is assumed that relevant physical quantities can be expanded as a double power series in terms of the bottom slope $z$ and the nonlinear parameter $\varepsilon$. It is assumed that the $x=O(1)$ following Chen et al. (2005, 2006). Thus, the particle displacements $x$ and $y$, the potential function $\phi$, wave pressure $P$, wave number $k$, and Lagrangian wave frequency $\sigma$ can be written as

$$x = x_0 + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c^m z^n [f_m(x_0,y_0,y_0) + f_m(x_0,y_0,y_0,0) t],$$  

(10)

$$y = y_0 + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c^m z^n [g_m(x_0,y_0,y_0) + g_m(x_0,y_0,y_0,0) t],$$  

(11)
where $S = \int k dx_0 - \sigma t$ is the phase function and $y(x_0, y_0, t)$ are the particle displacements and the Lagrangian variable $(x_0, y_0)$ are any two characteristic parameters, $\sigma$ is the nonlinear ordering parameter characterizing the wave steepness, and $M_{m,n,0}$ is the return flow. $\sigma(=2\pi T)$ is the angular frequency of the particle motion or the Lagrangian angular frequency for a particle reappearing at the same elevation, where $T$ is the period of particle motion. For a relatively gentle bottom slope $z$, it may be assumed that the $q$th differentiation of $A_{m,n,i}^i, A_{m,n,i}^i, B_{m,n,i}^i, B_{m,n,i}^i, a_{m,n,i}, a_{m,n,i}$, and $k_{m,n}$ with respect to $x_0$ are in the order of $z^q$ as

\[
\left( \frac{d^{q+k}A_{m,n,i}}{dx_0^q}, \frac{d^{q+k}A_{m,n,i}}{dx_0^q}, \frac{d^{q+k}B_{m,n,i}}{dx_0^q}, \frac{d^{q+k}B_{m,n,i}}{dx_0^q}, \frac{d^{q+k}a_{m,n,i}}{dx_0^q}, \frac{d^{q+k}a_{m,n,i}}{dx_0^q} \right) = O(z^q),
\]

$q, n = 0, 1, 2 \ldots N$

substituting Eqs. (10)–(15) into Eqs. (1)–(9), and collecting the terms of the same order in $\sigma$ and $z$, we obtain the necessary equations to each order of approximation. Then different order of $z(m), z(n)$ and harmonic($l$) may be separated, yielding a set of partial differential equations for each index $(m, n, l)$. Following these assumptions, the second-order asymptotic solutions for the problem can then be obtained by (Chen et al., 2012):

\[
S = \int (k) dx_0 - \sigma t = \int k_0, 0 dx_0 - \sigma t,
\]

\[
x(x_0, y_0, t) = x_0 + f_{1,0} + \phi f_{1,1} + \phi f_{1,2} + \phi f_{1,0},
\]

\[
y(x_0, y_0, t) = y_0 + g_{1,0} + \phi g_{1,1} + \phi g_{1,2}.
\]

\[
\phi = \phi_{1,0} + \phi_{1,1} + \phi_{1,2} + \phi_{1,0},
\]

\[
P = P_{1,0} + P_{1,1} + P_{1,2}.
\]

the detailed solutions are listed in Appendix A.

4. Results and discussions

4.1. The determination of the propagating velocity of the wave surface profile or the wave velocity $C_w$

Almost all the wave properties considered in this work can be directly found from the Lagrangian modeling results. The only unsolved property needing to be determined is the wave velocity $C_w(x_0, y_0=0, t)$ which is varying with the wave surface position. The wave velocity $C_w$ of the considered waves can be obtained below as shown in Fig. 3. Considering a surface particle marked with label $(x_0, y_0=0)$ is located at a point $A$ of the free surface with the horizontal coordinate $x(x_0, y_0=0, t) = x$ and the phase function $S = \int k dx_0 - \sigma t$ at time $t$. Along the propagating direction of the free surface, when time $t + dt$, $dt \rightarrow 0$, this point A with the wave velocity $C_w$ moves a horizontal distance $C_w dt$ to a new
Fig. 6. Comparisons between the dimensionless wave phase velocity on sloping bottom obtained by the nonlinear asymptotic solution and those from the experimental measurements data (circle point: experiment and solid line: the second-order Lagrangian solution).
position where an adjacent surface particle \( x_0 + dx_0, y = 0 \) travels from time \( t \) to \( t + dt \) and meet in that point. Thus, the horizontal coordinate of the new position of this point \( A \) at the free surface at time \( t + dt \) is \( x(x_0 + dx_0, y_0 = 0, t + dt) = x + dx \). According to the statement above, two necessary equations for determining the wave velocity \( C_w \) can be written in the remaining phase, where \( S \) is the constant, \( dt \to 0 \), and \( dx_0 \to 0 \), as follows:

\[
S = \int_{x_0}^{x_0 + dx_0} k(x, y = 0) dx_0 - \sigma(x_0, y = 0) dt
\]

\[
= \int_{x_0}^{x_0 + dx_0} k(x_0, y = 0) dx_0 - \sigma(x_0 + dx_0, y = 0)(t + dt)
\]  

(22)

\[ C_w dt = \frac{\sigma_w}{k(x_0)} dt = x(x_0 + dx_0, y_0 = 0, t + dt) \]

\[-x(x_0, y_0 = 0, t) = dx, dt \to 0, dx \to 0 \]

(23)

where \( \sigma_w = 2\pi/T_w^{1/2} \) is the wave period, \( k(x) \) is the wave number at the position

\[
x(x_0, y_0, t) = x_0 + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} e^{inx} A_{m,n}(x_0, y_0) F_{m,n} \int_{x_0}^{x_0 + dx_0} k(x_0, y_0) dx_0'
\]

\[-\sigma(x_0, y_0) F_{m,n}(x_0, y_0, \sigma t), \]

(24)

---

Fig. 7. (a-f) Comparisons between the dimensionless wave profile on the different water depths obtained by the second-order Lagrangian solution, the second-order Eulerian solution and those from the experimental measurements data (circle point: experimental data, solid line: the second-order Lagrangian nonlinear solution, and dashed line: the second-order Eulerian nonlinear solution (Chen et al., 2006). Case 1: \( h_0 = 0.09068 \text{ m} \) and \( T = 1.002 \text{ s} \).
the solution of Eqs. (22) and (23) are easily obtained via Eq. (24), which are
\[ kdx_0 = t \frac{\partial \sigma}{\partial x_0} dx_0 + \sigma dt + \text{H.O.T. or} \]
\[ dx_0 = \sigma dt / \left( k - t \frac{\partial \sigma}{\partial x_0} \right) = \sigma dt / k \left( 1 - t \frac{\partial \sigma}{k \partial x_0} \right), \quad y_0 = 0; \quad k = k(x_0) \]
and
\[ C_w = \frac{\sigma}{k(x_0)} = \left( \frac{\sigma}{k(x_0) - t \frac{\partial \sigma}{\partial x_0}} \right) + \sum_{m} \sum_{n} \frac{e^{m \beta \gamma}}{m \beta \gamma} \left[ \frac{\sigma}{k(x_0) - t \frac{\partial \sigma}{\partial x_0}} \left( \frac{\partial A_{m,n}}{\partial x_0} J_{m,n}(S) + \frac{\partial f_{m,n}}{\partial x_0} \right) \right] \]
\[ + \frac{\partial f_{m,n}}{\partial x_0}, \quad y_0 = 0 \]
respectively. The wave velocity \( C_w \) represented by (26) is related to the water depth, the wave steepness \( \varepsilon \), the bottom slope \( z \), the wave number \( k \), the angular frequency \( \sigma \), the phase function \( S \), and the mass transport of particle at the free surface. The result of the wave velocity \( C_w \) in Eq. (26) is consistent with that obtained by Chen et al. (2010) for the case of uniform water depth (i.e., \( z=0 \)). Hence, all the properties in the considered waves could be found using Eq. (26) given the wave period \( T_w \), the bottom slope \( z \), and the deep-water wave height \( H_0 \). Therefore, the wave velocity is to be conserved and completes the analysis to the considered waves directly in the Lagrangian framework up to the second-order solution. As shown in Fig. 4, the wave phase velocity decreases faster due to the large incident wave steepness. The latter implies that the breaking water depth increase with increasing incident wave steepness. Furthermore, the wave phase velocity decreases slowly while wave travel on a steep bottom slope (Fig. 5). Fig. 6 compares the measure wave phase velocities with Eq. (26). The analytical results are in agreement with the experimental data. Therefore, the present solution can predict the phase velocity very well.

4.2. Nonlinear wave transformation

It is well known that the surface profile of a breaking wave is not well prescribed by the Eulerian description (Chen et al., 2006), which is restricted to single valued functions of the spatial position and the dynamic boundary condition at the free surface which can only be satisfied in an approximate manner. This assumption limits the applicability of Eulerian description for breaking waves. As a wave shoals, in order to simulate the

![Fig. 8.](image)

Successive wave profiles prior to breaking plotted under three different bottom slopes (solid line: \( z=1/15 \), dashed line: \( z=1/10 \), and dotted line: \( z=1/5 \)).

![Fig. 9.](image)

Successive wave profiles prior to breaking plotted under three different wave steepnesses (solid line: \( k_0^0 H_0 = 0.1 \pi \), dashed line: \( k_0^0 H_0 = 0.06 \pi \) and dotted line: \( k_0^0 H_0 = 0.02 \pi \)).

![Fig. 10.](image)

Orbit of particle vary with four different bottom slopes (solid line: \( z=1/20 \), dashed line: \( z=1/15 \), dash-dotted line: \( z=1/10 \), and dotted line: \( z=1/5 \)).

![Fig. 11.](image)

Orbit of particle vary with four different wave steepnesses (solid line: \( k_0^0 H_0 = 0.1 \pi \), dashed line: \( k_0^0 H_0 = 0.08 \pi \), dash-dotted line: \( k_0^0 H_0 = 0.06 \pi \), and dotted line: \( k_0^0 H_0 = 0.04 \pi \)).
asymmetric waveform, the particle motion has to be transformed from the Eulerian to Lagrangian descriptions (Biesel, 1952; Chen et al., 2006). The free surface in the Lagrangian description is directly represented by an explicit parametric function of the particles and it allows greater flexibility for describing the actual shape and the wave kinematics above the mean water level. Moreover, the Lagrangian solution can exactly satisfy the dynamic boundary condition and ascertain zero pressure condition at the free surface motion. Hence, the Lagrangian description is more appropriate for limiting the free surface motion, and therefore we can expect that the solutions in the Lagrangian form can overcome some of the limitations from the classical Eulerian solutions (Biesel, 1952; Naciri and Mei, 1993; Chen et al., 2005, 2006; Buldakov et al., 2006).

Fig. 12. (a–l) Comparisons between the orbits of water particles obtained by the second-order Lagrangian solution and those from the experimental measurements of the PS motions on sloping bottom ((a–g) particle is on the surface, (h–l) particle is under the surface. Dotted line: experimental data and solid line: the second-order Lagrangian solution) (the wave conditions were listed in Table 3).
As the wave height reaches its highest limit, the crest is fully developed as a summit which can be calculated as the spatial surface profile by a system of Lagrangian coordinates. In this approach, the new displacement components of water particle ($x$ and $y$) to the second order approximation have been obtained (see Eqs. (18) and (19)). The wave elevation varies from deep water to the breaking wave location and the Eulerian and Lagrangian approximations can be evaluated against laboratory observations (see Fig. 7a–f). Fig. 7 shows that the wave elevations are higher and steeper from deep water to shallow water. Furthermore, Fig. 7d shows that the Eulerian result starts to become wider than the experimental data, even in the Eulerian solution has the secondary wave occurs in Fig. 7e and f. On the other hand, the Lagrangian nonlinear solution is better than Eulerian nonlinear solution to predict waves the evaluation on shallow water. Review Fig. 8, when wave propagates on a gentle sloping bottom, wave profile will evolve to asymmetric and will tend...
to plunging breaker (Elgar and Guza, 1985). The breaking water depth increases with incident wave steepness. The results of Figs. 8 and 9 suggest that the wave breaking type and breaking water depth are influenced by both the wave steepness and bottom slope.

4.3. Particle trajectory for progressive wave on a sloping bottom

The Lagrangian solution for water-particle displacement can be employed in order to predict the water particle motion beneath the wave passage. The parametric functions for the water particle at any location in Lagrangian coordinates (x, y) are given by Eqs. (18) and (19). From Fig. 10, it shown that the orbit of particle tends to the elliptic shapes for the milder sloping bottom. Furthermore, the mass transport is larger due to large incident wave steepness as shown in Fig. 11. Fig. 12 shows the analytical and experimental trajectories of a progressive wave over a sloping bottom near the breaking point, where $k_{0,0X}$ and $k_{0,0Y}$ are the dimensionless position of PS particle in the x- and y-directions, respectively. The particles do not move in closed orbital motion owing to the second-order mass transport velocity, which decreases exponentially with the water depth. For surface particles, the trajectory is more symmetric in deep water than in shallow water (see Fig. 12a–g). Under the surface, the trajectories become elliptic since the vertical excursion of the particle is

---

**Fig. 13.** Relationship between breaking wave steepness $H_b/L_0$ and relative breaking depth $d_b/L_0$ in different bottom slopes ($\alpha = 1/20$, dashed line: $\alpha = 1/10$, dash-dotted line: $\alpha = 1/5$, and dotted line: $\alpha = 1/3$).

**Fig. 14.** Relationship between the dimensionless breaking wave height and incident wave steepness in different bottom slopes ($\alpha = 1/20$, dashed line: $\alpha = 1/15$, dash-dotted line: $\alpha = 1/10$, and dotted line: $\alpha = 1/5$).

**Fig. 15.** (a–c) Comparison of the breaking criteria under three bottom slopes ($\alpha = 1/10$, 1/5 and 1/3).
smaller than its horizontal excursion unlike the trajectories near the mean water level (Fig. 12h–l). In addition, the particle trajectories tend to orientated to be parallel to the bottom slope due to the bottom slope effects on wave velocity.

4.4. Wave breaking index

The increase in wave height owing to shoaling becomes depth-limited as the wave propagates into shallow waters. The celerity is reduced and hence the particle velocity of the wave crest is faster than the wave celerity inducing the wave breaking. Wave breaking is controlled by wave steepness, wave height, bottom slope, and other factors. Therefore, the breaking phenomenon is very complex and most of the previous studies gave breaker characteristics only based on experimental studies, empirical, or semi-empirical formulas calibrated against laboratory data. On the other hand, the present approach is the only one that gives an analytical framework of the breaking wave process. In order to describe the breaking wave mechanism, a breaking criterion needs to be implemented given by

\[ u/C_w = 1 \]

where \( C_w \) is wave phase velocity and \( u \) is the horizontal velocity of particle at the wave crest. Up to the wave breaking, the horizontal velocity of the particle on the surface, \( u = \partial(x(0,0,t))/\partial t = C_w \) can be obtained as (up to \( O(\epsilon^2, \epsilon^3, \epsilon^4, \ldots) \)): when

\[
u = x_t - C_w = k_0[0] + \frac{1}{2} B_{1,0,1} \frac{k_0 \cos \delta + d}{\sinh k_0 d} \cos 2S,
\]

\[
= 1,
\]

the maximum surface elevation which appears as a wave breaks is given by

\[
\frac{d\eta}{dx_0} = \frac{dy(x_0,0,t)}{dx_0} = \frac{dy(x_0,0,t)}{dS} \frac{dS}{dx_0} = \left( \frac{dy}{dS} \frac{dS}{dx_0} \right) \Bigg|_{x_0=0} = 0
\]

and we have

\[
\frac{dy}{dS} = -B_{1,0,1} \frac{k_0 \sinh k_0 d}{\sinh k_0 d} \sin \delta + 3 \frac{B_{1,0,1} k_0}{4} \frac{\sinh 2k_0 d}{\sinh k_0 d} \sin 2S = 0
\]

from (27) and (28), the phase angle of the breaking wave \( \theta = S_0 \) and the breaking wave height can be obtained. Comparing to the gentle bottom slope, the wave breaking point location is farther away from the shore when wave propagates on steeper bottom slope. In the same depth of breaking wave, the steeper the bottom slope will occur the sharper steepness of breaking wave; on the other side, in the same steepness of incident wave, the steeper the bottom slope will occur the sharper steepness of breaking wave (see Figs. 13 and 14).

We present a comparison given by the present theory against the empirical formula of Goda (2010) and the experiments made by Iwagaki et al. (1974), Deo and Jagdale (2003) and Tsai et al. (2005) in Fig. 15a–c. It is found that the analytical breaking index is in good agreement with the experimental results for three different slopes. However, for the steep slope of 1/3 shown in Fig. 15c, the result of Goda’s empirical formula overpredicted the experimental observations and the analytical results. The accuracy of present result in mild slope needs further improvement to include the effects of higher order wave steepness on a sloping bottom and the effect of bottom friction in shallow water.

Based on the results depicted in Fig. 16a–c, it showed that the analytical results are in better agreement with the experimental data (Iwagaki et al., 1974; Deo and Jagdale, 2003; Tsai et al., 2005) than empirical formulations presented the breaking wave height in terms of deep water wave conditions (e.g., Sunamura, 1983 and in Rattanapitikon and Shibayama, 2000). For instance, Sunamura’s empirical formula always overpredicted the measured values. Therefore, it is easy to find the fact that other scholars’ empirical formula did not conform to the experimental results on steep slopes. It is referred by Tsai et al. (2005). However, the present analytical result present a satisfactorily agreement on steep slopes.

**Fig. 15.** (a–c) Comparison the analytical results and the empirical formulas with the experiments for the breaking wave height under three bottom slopes (\( z = 1/10, 1/5 \) and 1/3).
5. Conclusions

Laboratory experiments are conducted in a wave flume in order to determine wave orbital motions and wave phase speed variation for monochromatic waves propagating on a sloping bottom. Furthermore, a Lagrangian description for water-particle motion of second-order asymptotic solutions in explicit form which includes parametric functions is presented. These explicit expressions enable a consistent description of wave shoaling from deep to shallow water. The solutions also provide information for the process of successive deformation of a wave profile, water particle trajectory, and higher order wave statistics.

Model-data comparison revealed that the Lagrangian second-order solution can describe the nonlinear wave transformation on the sloping bottom better than the Eulerian second-order solution. Furthermore, the cross-shore variation of phase velocity and orbital motion are also well described by the Lagrangian second-order solution. Therefore, the breaking characteristics (i.e., wave height and breaking depth) can be easily predicted assuming a threshold in the relationship between the wave celerity and the maximum orbital velocity.

While the purpose of the present study is to establish a rational mathematical model, which could predict the wave breaking index, its accuracy was verified by experimental data. The model predicts the breaking even for the steep slope cases. On the other hand, the model accuracy for predicting wave breaking on the mild slopes can be further improved by including the effects of higher order wave steepness on a sloping bottom, bottom friction in shallow water, and/or by employing a different value of the wave breaking criterion u/Cωo.

Acknowledgments

The work was supported by the Research Grant Council of the National Science Center, Taiwan, through Project nos. NSC99-2923-E-110-001-MY3, NSC99-2221-E-110-087-MY3 and NSC100-2917-I-110-002.

Appendix A. The nonlinear asymptotic solutions up to second order

The ε^1/2 order solutions as follows:

\[ f_{1,0} = A_{1,0,1}(x_0) \cos k_0_0(0 + d) \sin S, \]
\[ g_{1,0} = B_{1,0,1}(x_0) \sin k_0_0(0 + d) \cos S, \]
\[ A_{1,0,1} = -B_{1,0,1} f_{1,0} = G_{1,0} = \sigma_{0,0} = \sigma_{0,0,0} = \phi'_{1,0} = 0, \]
\[ \phi_{1,0} = x_{1,0}(x_0) \cos \left( k_0_0(0 + d) \right) \sin S, \]
\[ \sigma_{0,0,0} = g_{0,0} \tan \left( k_0_0(0 + d) \right), \]
(A-1)

the ε^1/2 order solutions as follows:

\[ A_{1,1,1} = B_{1,0,1} \left\{ \frac{k_0_0(0 + d)^2}{\sin k_0_0(0 + d)} - k_0_0(0 + d) \frac{1}{D \tan \left( k_0_0(0 + d) \right)} \right\} \cos k_0_0(0 + d) = \left\{ \frac{k_0_0(0 + d)^2}{\sin k_0_0(0 + d)} - k_0_0(0 + d) \frac{1}{D \tan \left( k_0_0(0 + d) \right)} \right\} \sinh k_0_0(0 + d), \]
\[ B_{1,1,1} = B_{1,0,1} \left\{ \frac{k_0_0(0 + d)^2}{\sin k_0_0(0 + d)} - k_0_0(0 + d) \frac{1}{D \tan \left( k_0_0(0 + d) \right)} \right\} \sin k_0_0(0 + d) = \left\{ \frac{k_0_0(0 + d)^2}{\sin k_0_0(0 + d)} - k_0_0(0 + d) \frac{1}{D \tan \left( k_0_0(0 + d) \right)} \right\} \cosh k_0_0(0 + d), \]
\[ \phi_{1,1,1} = -g_{0,0} \cos \frac{k_0_0(0 + d)^2}{\sin k_0_0(0 + d)} - k_0_0(0 + d) \frac{1}{D \tan \left( k_0_0(0 + d) \right)} \right\} \cos k_0_0(0 + d) = \left\{ \frac{k_0_0(0 + d)^2}{\sin k_0_0(0 + d)} - k_0_0(0 + d) \frac{1}{D \tan \left( k_0_0(0 + d) \right)} \right\} \sinh k_0_0(0 + d), \]
\[ k_{0,1} = \sigma_{0,1} = 0; B_{1,0,1} = a = \frac{a}{k_0_0 \cot \left( k_0_0(0 + d) \right)}, \]
(A-2)

References